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ATTRACTION FOR AUTONOMOUS MECHANICAL SYSTEMS WITH SLIDING FRICTION†

V. M. MATROSOV and I. A. FINOGENKO

Moscow and Irkutsk

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Issues on attraction in autonomous mechanical systems with ideal holonomic bilateral constraints acted upon by potential gyroscopic dissipative forces and forces of sliding friction are considered. In particular, the semi-invariance of ω-limit sets and the conditions for the dichotomy of such systems are established. The investigation is based on the invariance principle using several Lyapunov functions, combining the methods of [1] with the La Salle invariance principle [2, 3] applied to autonomous systems with a discontinuous right-hand side. © 1998 Elsevier Science Ltd. All rights reserved.

Investigations of the dynamical properties of mechanical systems with sliding friction based on the theory of ordinary differential equations essentially involve constructing and developing a general mathematical theory of such equations. Two types of difficulties arise: the first is due to the possibility that the condition that all the motions shall exist, be unique and be continuable without limit is violated (within the framework of the accepted mechanical model of the system, including the assignment of the friction forces in accordance with Coulomb's law); the second is associated with discontinuities of the generalized accelerations at points of relative rest (with discontinuous right-hand sides of the equations of motion), and make it impossible to use well-developed methods of the classical theory of differential equations for systems with friction.

The "non-uniqueness" or "impossibility" of motions, which was first pointed out by Painlevé [4], has been the subject of numerous investigations (see [5–10]‡). The current publications on this topic are mainly either discursive in character or involve both introducing and taking into account an additional mechanical hypothesis, studying the conditions under which Painlevé paradoxes appear and analysing the regions where they manifest themselves.

It is worth noting that even if the equations of motion are uniquely defined and consistent with Coulomb's laws (in which case there are no Painlevé paradoxes), difficulties still arise in investigating these equations due to the discontinuous dependence of the friction forces on the generalized velocities.

Below we investigate certain properties of the motions (semi-invariance of ω-limit sets and the attraction of stagnant zones) of mechanical systems with sliding friction that were introduced and investigated in [11, 12]. The results are obtained using the mathematical theory of systems with friction, developed in [11–14].

We will first establish a modified principle of invariance for autonomous equations of a general form, which will enable us to define the conditions of attraction of motions of the system by a certain set M using a set of auxiliary Lyapunov functions.

1. THE PRINCIPLE OF INVARIANCE AND ATTRACTION FOR AUTONOMOUS SYSTEMS

Consider the autonomous differential equation

$$\dot{x} = f(x) \tag{1.1}$$

with the function $f: \Omega \to R^n$ defined in some region $\Omega \subset R^n$. By a right-sided solution of Eq. (1.1) on $[t_0, t_1)$ with initial conditions (t_0, x_0) we mean a continuous function $x(t)$, which is differentiable from the right and defined in the interval $[t_0, t_1)$, and satisfies the relations

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‡See also LE SUAN AN, Theory of mechanical systems with sliding friction. Unpublished paper. Vsesoyuz. Inst. Nauch. Tekh. Inf., No. 84-V87.

$$D^+x(t) = f(x(t)), \quad x(t_0) = x_0$$

for all $t \in [t_0, t_1)$, where $D^+x(t)$ is the right derivative of the function $x(t)$.

Everywhere below it is assumed that the solutions of Eq. (1.1) are right-sided, $t_0 = 0$, and the following basic conditions are satisfied for Eq. (1.1).

1. For any initial state $x_0 \in \Omega$, a local solution of Eq. (1.1) exists.
2. The function $f(x)$ is locally bounded.
3. The limit $x(t)$ of any sequence of solutions of Eq. (1.1) which converges uniformly in $[t_0, t_1)$ is a solution of Eq. (1.1), provided that $x(t) \in \Omega$ for all $t \in [0, t_1)$.

Uniqueness or right-sided uniqueness of the solutions is not assumed. Each of the conditions 1–3 is obviously satisfied if f is continuous; conditions under which they are satisfied even for systems with friction are given in [12–14].

The ω-limit set of a solution $x(t)$ of Eq. (1.1) defined in the maximum right-hand interval of existence $[0, \omega)$ is denoted by $\Lambda^+(x)$. We will use the same definitions of semi-invariance of sets, attraction and weak attraction as in [3].

Lemma 1.1. Any solution of Eq. (1.1) can be continued to the maximum right-hand interval of existence $[0, \omega)$, where either $\omega = +\infty$ or $\omega < +\infty$ and for any compact set $K \subset \Omega$ there is a point $t_K \in (0, \omega)$ such that $x(t) \notin K$ for all $t \in (t_K, \omega)$ ($x(t)$ tends to the boundary $\partial\Omega$ of the set Ω).

In particular, if $\Lambda^+(x) \cap \Omega \neq \varnothing$, then $\omega = +\infty$.

Proof. The existence of a continuation of the solution $x(t)$ to the maximum right-hand interval of existence can be established using Zorn's lemma and basic conditions 1 and 2 above.

To prove that $x(t)$ tends to the boundary of the set Ω, assume otherwise. Then there is a compact set $K \subset \Omega$, and sequences of points $t_i \to \omega$, $x(t_i) \to a$ for which $x(t_i) \in K$ and $a \in K$. By an argument similar to that in [15 p. 26], it can be shown that $\lim_{t\to\omega} x(t) = a$. But then the solution $x(t)$ can be continued to the right, which is impossible.

For completeness, we will state some properties of ω-limit sets which are true for any continuous curves: $\Lambda^+(x)$ is a closed set; $\Lambda^+(x) \neq \varnothing$ if, and only if, the function $\|x(t)\|$ is not infinitely large as $t \to \omega$; the set $\Lambda^+(x)$ is bounded if, and only if, the function $x(t)$ is bounded and $\lim_{t\to\omega} d(x(t), \Lambda^+(x)) = 0$ where d is the distance from the point to the set.

Lemma 1.2. The set $\Delta \triangleq \Delta^+(x) \cap \Omega$ is semi-invariant.

Proof. Let $a \in \Lambda \neq \varnothing$. Then, according to Lemma 1.1, $\omega = +\infty$ and there is a sequence of points $t_k \to +\infty$ such that $t_{k+1} - t_k > 0$ for some $\alpha > 0$ and $\lim_{k\to\infty} x(t_k) = a$.

It is obvious that the functions $y_k(t) = x(t_k + t)$, the solutions of Eq. (1.1), are defined in the interval $[0, \alpha)$ and $y_k(0) = x(t_k) \to a$. Since the function f is locally bounded, starting from some index k the sequence $\{y_k(t)\}_1^\infty$ is uniformly bounded and uniformly continuous in some interval $[0, t_1]$, $0 < t_1 < \alpha$ and the functions $y_k(t)$ take values in the neighbourhood of the point a belonging to the set Ω together with its closure. Thus, by Artsel's theorem, there is a subsequence of this sequence which is uniformly convergent in $[0, t_1]$ and the limit $y(t)$ of which will be a solution of Eq. (1.1), defined in $[0, t_1]$, with initial condition $y(0) = a$. Obviously, $y(t) \in \Lambda$ for all $t \in [0, t_1)$.

Thus, for any $a \in \Lambda$ there is a solution $y(t)$ of Eq. (1.1) which is defined in some interval $[0, t_1]$ and for which $y(t) \in \Lambda$ for all $t \in [0, t_1)$. Thus for any $a \in \Lambda$ there is a solution $y(t)$ of Eq. (1.1) with initial condition $y(0) = a$ which cannot be continued in the set Λ, that is, $y(t) \in \Lambda$ for all $t \in [0, \omega_y)$ and there is no solution $z(t)$ of Eq. (1.1) which is the same as $y(t)$ in the interval $[0, \omega_y)$ and such that $z(t) \in \Lambda$ for all $t \in [0, \omega_z)$, where $\omega_z > \omega_y$.

If ω_y is not the right-hand end of the maximum interval of existence of the solution $y(t)$ (in the set Ω), then $\lim_{t\to\omega_y} y(t) \in \Lambda$ exists and then $y(t)$ can be continued in the set Λ, which is impossible. Hence, the solution $y(t)$ cannot be continued. This proves that the set Λ is semi-invariant.

Note 1.1. If $\Lambda^+(x) \subset \Omega$ is a bounded set, for any $a \in \Lambda^+(x)$ there is a solution $y(t)$ of Eq. (1.1) which is invariant with respect to $\Lambda^+(x)$ and defined in $(-\infty, +\infty)$ and satisfies the condition $y(0) = a$.

Let $w(x)$ denote an arbitrary function with non-negative values, defined in the region Ω. We put $E(w = 0) \triangleq \{x \in \Omega{:}w(x) = 0\}$.

For a locally Lipschitz function $V{:}\,\Omega \to R^n$, by virtue of Eq. (1.1) we will denote the right-hand upper Dini derivative by $D^{*+}V(x)$. It can be computed using Yoshizawa's theorem (cf. [3, p. 269]) which, as is easily verified, also holds for the systems discussed here.

By Lemma 1.2, with the given assumptions (under the basic conditions) the La Salle invariance principle applies to Eq. (1.1) for autonomous systems, as stated in the theorems of [2, 3]. It will be convenient to put these in a different form, which can be proved as in [3, p. 190].

Theorem 1.1. Let $x(t)$ be a non-continuable solution of Eq. (1.1) and $V : \Omega \to R^n$ a locally Lipschitz function for which $D^{*+}V(x) \leqslant -w(x)$ on $[0, \omega)$. Then

$$\Lambda^+(x) \cap \Omega \subset E(w=0)$$

Theorem 1.2. Suppose that for Eq. (1.1) and some set $M \subset \Omega$ there is a finite set of locally Lipschitz functions $V_i(x)$, $0 \leqslant i \leqslant N$ for which $D^{*+}V_0(x) \leqslant -w(x)$ for all $x \in \Omega$ and that for any $x \in E(w = 0)\backslash M$ there is a function V_i, $1 \leqslant i \leqslant N$ for which $V_i(x) = 0$ and $D^{*+}V_i(x) \neq 0$.

Then for any non-continuable solution $x(t)$ of Eq. (1.1), we have

$$\Lambda^+(x) \cap \Omega \subset \overline{M} \tag{1.2}$$

Here

$$\Lambda^+(x) \subset \overline{M} \cup \partial\Omega \tag{1.3}$$

and the following assertions are true:

1. either $\|x(t)\| \to +\infty$ or $x(t)$ tends weakly to the set $M \cup \partial\Omega$ as $t \to \omega$;
2. either $x(t)$ is unbounded, or $x(t)$ tends to the set $M \cup \partial\Omega$ as $t \to \omega$;
3. if $M \cup \partial\Omega = \varnothing$, then $\|x(t)\| \to +\infty$ as $t \to \omega$.

Proof. The inclusion (1.3) follows from (1.2) and Lemma 1.1. Parts 1–3 of the theorem follow from (1.3). Thus if we can establish that condition (1.2) holds, the theorem will have been proved.

The case $\Lambda^+(x) = \varnothing$ or $\Lambda^+(x) \subset \partial\Omega$ is trivial. Let $\Lambda^+(x) \cap \Omega \neq \varnothing$ and $a \in \Lambda^+(x) \cap \Omega$. Then $\omega = +\infty$ and there is a sequence of points $t_k \to +\infty$ for which $\lim_{k\to\infty} x(t_k) = a$.

Suppose that $a \notin \overline{M}$. By virtue of Lemma 1.2 and Theorem 1.1, there is a solution $y(t)$ of Eq. (1.1) for which $y(0) = a$ and $y(t) \in E(w = 0)$ for all $t \in [0, \omega_y)$. Since the set $\overline{M}$ is closed, $y(t) \notin \overline{M}$ for all $t \in [0, \alpha)$ for some $\alpha \in [0, \omega_y)$.

It follows from the conditions of the theorem that there is a function V_{i_1} and a sufficiently small number $h_1 > 0$ for which $V_{i_1}(a) = 0$, $V_{i_1}(y(h_1)) \neq 0$ and $(y(h_1)) \neq \overline{M}$. Hence there is a function V_{i_2}, $i_1 \neq i_2$ for which $V_{i_2}(y(h_1)) = 0$, $V_{i_2}(y(h_1 + h_2)) \neq 0$ and $y(h_1 + h_2) \notin \overline{M}$ for some $h_2 > 0$ so small that it is also true that $V_{i_1}(y(h_1 + h_2)) \neq 0$. Continuing this process, we obtain a point $t_N = \Sigma_{i=1}^N h_i$ such that $t_N \in [0, \alpha)$, $V_i(y(t_N)) \neq 0$ for all $i = 1, \ldots, N$, $y(t_N) \in E(w = 0)\backslash M$. This contradicts the conditions of the theorem, and so (1.2) holds. This proves the theorem.

For each $x \in \Omega$, we will put

$$\underline{w}(x) = \begin{cases} w(x) & , x \in \Omega \\ \underline{\lim}_{x' \to x, x' \in \Omega} w(x') & , x \in \partial\Omega \end{cases}$$

We will say that the function $V(x)$ is continuous up to the boundary if for each point $x \in \partial\Omega$ there is a finite limit $\lim_{x' \to x, x' \in \Omega} V(x')$. The function f will be said to be locally bounded on the boundary if for any point $x \in \partial\Omega$ the function f is bounded on the intersection of some neighbourhood of x and the set Ω.

Theorem 1.3. Let $V_0(x)$ be a locally Lipschitz function which is continuous up to the boundary and for which

$$D^{*+}V_0(x) \leqslant -w(x) \tag{1.4}$$

for all $x \in \Omega$, and let the function f be locally bounded on the boundary. Then $\Lambda^+(x) \subset E(\underline{w} = 0)$ for any solution of Eq. (1.1) which is defined for all $t \geqslant 0$.

Proof. Let $a \in \Lambda^+(x)$. If $a \in \Omega$, the theorem follows from Theorem 1.1.

Suppose that $a \in \partial\Omega$ and $\lim_{tk\to+\infty} x(t_k) = a$. It follows from the conditions of the theorem that there is a finite limit $\lim_{tk\to+\infty} V(x(t)) = c$.

Let $a \notin E(\underline{w} = 0)$. Then there are number $\delta > 0$ and $\alpha > 0$ such that $w(x') > \alpha$ for all $x' \in S_\delta(a) \cap \Omega$, where $S_\delta \triangleq \{x':\|x - x'\| < \delta\}$. Without loss of generality, for the sequence of points t_k it can be assumed that $t_{k+1} - t_k > h > 0$ for all $k = 1, 2, \ldots$. Since the function f is locally bounded on the boundary, there are numbers $h_0 \in (0, h)$ and k_0 such that $x(t_k + t) \in S_\delta(a) \cap \Omega$ for all $t \in [0, h_0)$ and $k \geqslant k_0$. But then

$$V(x(t)) \leqslant V(x(0)) - \alpha(k - k_0)h_0$$

provided that $k > k_0$, $t > t_{k+1}$. This last inequality contradicts the fact that the function $V(x(t))$ is bounded. Hence $a \in E(\underline{w} = 0)$. This proves the theorem.

Theorem 1.4. Let $M \subset \Omega$ be some set and $V_0(x)$ a locally Lipschitz function which is continuous up to the boundary and satisfies inequality (1.4). We will assume that the function f is locally bounded on the boundary and that in some neighbourhood of the set $\bar{\Omega}$ continuously differentiable functions $V_i(x)$ $(1 \leqslant i \leqslant N)$ with the following property are defined: for any $x \in \underline{E(\underline{w} = 0)\backslash \bar{M}}$ there is a function V_i such that $V_i(x) = 0$ and for which $D^+V_i(x) \neq 0$ if $x \in (E(w = 0) \cap \Omega\backslash \bar{M}$

$$(\text{grad } V_i(x), \bar{f}(x)) > 0 \text{ for all } \bar{f}(x), \text{ if } x \in (E(\underline{w} = 0) \cap \Omega)\backslash \bar{M} \tag{1.5}$$

where $\bar{f}(x)$ are the limiting values of the function f at the point x.

Then for any solution $x(t)$ of Eq. (1.1), which is defined for all $t \geqslant 0$, we have the inclusion

$$\Lambda^+(x) \subset \bar{M} \tag{1.6}$$

Here

1. either $\|x(t)\| \to +\infty$, or $x(t)$ weakly tends to the set M as $t \to +\infty$;
2. either the function $x(t)$ is unbounded or $x(t)$ tends to the set M as $t \to +\infty$;
3. if $M = \varnothing$, then $\|x(t)\| \to +\infty$ as $t \to +\infty$ for any solution of Eq. (1.1).

Proof. By Theorem 1.2, in order to prove (1.6) it is sufficient to show that $\Lambda^+(x) \cap \partial\Omega \subset \bar{M}$. Suppose otherwise, i.e. suppose $a \in (\Lambda^+(x) \cap \partial\Omega)\backslash \bar{M}$ exists.

Let $x(t_k) \to a$, where $t \to +\infty$, $t_{k+1} - t_k > h > 0$ for $k = 1, 2, \ldots$. Since the function f is locally bounded on the boundary, the Artsel–Ascoli theorem shows easily that there is a sufficiently small number $h_0 \in (0, h)$ for which there is a subsequence of the sequence of solutions $y_k(t) = x(t_k + t)$, $t \in [0, h)$ of Eq. (1.1) which converges uniformly in $[0, h_0]$. Let its limit be denoted by $y(t)$.

It is obvious that $y(0) = a$ and $y(t) \in \Lambda^+(x)$ for all $t \in [0, h_0)$. We will show that $y(t) \in \partial\Omega$ for all $t \in [0, \alpha)$ for some $\alpha \in [0, h_0)$. Assuming the contrary, we find that there is a sequence of points $t_i \to +0$, $t_i \neq 0$ for which $y(t_i) \notin \partial\Omega$ for all $i = 1, 2, \ldots$. Then according to Theorem 1.2 $y(t_i) \in \bar{M}$ and, therefore, $y(0) \in \bar{M}$, which is impossible. From this contradiction and Theorem 1.3 we have

$$y(t) \in (E(\underline{w} = 0) \cap \partial\Omega) \setminus \bar{M}$$

for all $t \in [0, \alpha)$ for some small $\alpha > 0$.

According to [16, p. 53 Lemma 1, p. 60 Corollary 1 and p. 56 Theorem 1]

$$\text{cont } y(t) \subset F(y(t)) \tag{1.7}$$

for all $t \in [0, \alpha)$, where $F(x)$ is the convex envelope of all the limiting values of the function f at each point $x \in \bar{\Omega}$ and cont $y(t)$ is the contingency of the function $y(t)$.

Let V_{i_1} be a function whose existence is assumed under the conditions of the theorem for the point $y(0)$. Then $V_{i_1}(y(0)) = 0$ and from (1.5) we have

$$\sup\{(\text{grad } V_{i_1}(y(0)), z) : z \in F(y(0))\} > 0 \tag{1.8}$$

It follows from (1.7) and (1.8) that any right derivative Dini number of the function V_{i_1} along $y(t)$ at the point $t = 0$ is non-zero. Thus there is a sufficiently small number $h_1 \in [0, \alpha)$ such that $V_{i_1}(y(h_1)) \neq 0, y(h_1) \notin \bar{M}$. Then choosing the function V_{i_2} such that $V_{i_2}(y(h_1)) = 0$, we can see that there is a number $h_2 > 0$ such that $V_{i_2}(y(h_1 + h_2)) \neq 0$, $V_{i_1}(y(h_1 + h_2)) \neq 0, y(h_1 + h_2) \notin \bar{M}$. The proof continues as in Theorem 1.2.

2. ATTRACTION IN SYSTEMS WITH FRICTION

We will now consider a mechanical system with k degrees of freedom, under ideal constraints which are holonomic and time-independent, with the forces of sliding friction added to the active forces.

Its equations of motion can be written in Laplace form [11]

$$\frac{d}{dt}\frac{\partial T_a}{\partial \dot{q}^i} - \frac{\partial T_a}{\partial q^i} = Q_i^T + Q_i^A, \quad i = 1, \ldots, k \tag{2.1}$$

Here T_a is the kinetic energy of the system, representing the sum $T_a = T + T_1 + T_0$ of the positive definite quadratic form T of generalized velocities with the symmetric matrix $A(q) = [a_{ij}(q)]_1^k$, the linear form of the generalized velocities $T_1 = \Sigma_{i=1}^k a_i(q)\dot{q}^i$, and the function $T_0(q)$. Let the generalized forces of sliding friction have the form

$$Q_s^T(q, \dot{q}, \ddot{q}) = \begin{cases} -f_s(q^s, \dot{q}^s)|N_s(q, \dot{q}, \ddot{q})|\operatorname{sgn}\dot{q}^s, \text{ if } \dot{q}^s \neq 0 \\ f_s(q^s, 0)|N_s(q, \dot{q}, \ddot{q})|\operatorname{sgn} Q_s^{T0}(q, \dot{q}, \ddot{q}), \text{ if } \dot{q}^s = 0, \\ \qquad |Q_s^{T0}(q, \dot{q}, \ddot{q})| > f_s(q^s, 0)|N_s(q, \dot{q}, \ddot{q})|_{\dot{q}^s=0} \\ Q_s^{T0}(q, \dot{q}, \ddot{q}), \qquad\qquad \text{if } \dot{q}^s = 0, \\ \qquad |Q_s^{T0}(q, \dot{q}, \ddot{q})| \leqslant f_s(q^s, 0)|N_s(q, \dot{q}, \ddot{q})|_{\dot{q}^s=0} \end{cases}$$

$$Q_s^{T0}(q, \dot{q}, \ddot{q}) \triangleq \sum_{j=1, j\neq s}^{k} a_{s,j}(q)\ddot{q}^j - [g_s(q, \dot{q}) + Q_s^A(q, \dot{q})]_{\ddot{q}=0}$$

where $1 \leqslant s \leqslant k_*$, $k_* \leqslant k$, $f_s(q^s, \dot{q}^s) > 0$ are the friction coefficients, $|N_s(q, \dot{q}, \ddot{q})|$ are the moduli of the normal reactions, and $Q_s^{T0}(q, \dot{q}, \ddot{q})$ are the friction forces at relative rest. For $s = k_* + 1, \ldots, k$ we assume $f_s = 0$.

It can be assumed here, for example, that

$$Q_s^A(q, \dot{q}) = D_s(q, \dot{q}) + K_s(q)$$

$$\Gamma_s(q, \dot{q}) \triangleq \frac{\partial T_1}{\partial q^s} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}^s} = \sum_{j=1}^{k}\left(\frac{\partial a_j}{\partial q^s} - \frac{\partial a_s}{\partial q^j}\right)\dot{q}^j, \quad Q_s^e(q) \triangleq \frac{\partial T_0}{\partial q^s}$$

where $D_s(q, \dot{q})$ are dissipative forces, $K_s(q) = -\partial\Pi/\partial q^s$, $\Pi(q)$ is the potential energy of the system, $\Gamma_s(q, \dot{q})$ are gyroscopic forces with the conditions $\Gamma_s(q, 0) \equiv 0$; $D_s(q, 0)$ and $Q_s^e(q)$ are generalized transfer forces of inertia ($s = 1, \ldots, k$). Then (2.1) can be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = D_i + \Gamma_i + K_i + Q_i^e + Q_i^T, \quad i = 1, \ldots, k$$

and the function g_s in the expression for the force of static friction is defined as

$$g_s(q, \dot{q}) = \Gamma_s(q, \dot{q}) + Q_s^e(q) + \frac{1}{2}\sum_{\nu=1}^{k}\sum_{j=1}^{k}\frac{\partial a_{\nu j}}{\partial q^s}\dot{q}^\nu\dot{q}^j - \sum_{j=1}^{k}\sum_{\nu=1}^{k}\frac{\partial a_{sj}}{\partial q^\nu}\dot{q}^\nu\dot{q}^j$$

Under certain conditions (cf. [14]) Eqs (2.1) are solvable for the generalized accelerations and become

$$\ddot{q} = G(q, \dot{q}) \tag{2.2}$$

with the discontinuous function G. Theorems of existence of right-sided solutions of (2.2) have been proved and their properties studied. In particular, all the properties 1–3 of Section 1 of the basic conditions have been established [14, Lemma 2, Lemma 5 and Theorem 1]. Accordingly, we will take Eqs (2.1) and (2.2) to be equivalent and assume that $\Omega = R^{2k}$.

Let $V_0 \triangleq \Pi + T - T_0$. Multiplying Eqs (2.1) by $\dot q^i$ and then adding them, we obtain the equation of energy dissipation

$$D^+V_0(q,\dot q) = -\sum_{i=1}^{k_*} f_i\,|N_i|\,|\dot q^i| + \sum_{i=1}^{k} D_i(q,\dot q)\dot q^i (\leqslant 0)$$

Let $J \subset (1,\dots,k_*)$. Defining

$$w_J(q,\dot q) = \sum_{i\in J} f_i\,|N_i|\,|\dot q^i|,\quad H_J = \{(q,\dot q): \dot q^i = 0, i\in J\}$$

$$M_J = \{(q,\dot q): \dot q^i = 0,\ f_i\,|N_i| \geqslant |Q_i^{T0}|,\ i\in J\}$$

$$M = \{(q,0): f_i\,|N_i| \geqslant |K_i + Q_i^e|,\ i = 1,\dots,k_*;\ K_i + Q_i^e = 0,\ i = k_*+1,\dots,k\}$$

Obviously, if $J' \subset J$, then

$$w_{J'} \leqslant w_J \leqslant -D^+V_0,\quad H_J \subset H_{J'},\quad M_J \subset M_{J'}$$

and $M \subset M_J \subset H_J \subset E(w_J = 0)$ always. The set M is the set of equilibrium positions for Eqs (2.1). The sets H_j and M are closed.

As in [11, 12], we put $N(\dot q) = \{i \in (1,\dots,k_*): \dot q^i = 0\}$. It is easily verified that $(q,\dot q) \in H_j \Leftrightarrow J \subset N(\dot q)$ and

$$((q,\dot q) \in E(w_J = 0) \setminus H_J) \Leftrightarrow (J \setminus N(\dot q) \neq \emptyset) \wedge (\forall i \in J \setminus N(\dot q), |N_i| = 0) \tag{2.3}$$

It follows immediately from our assumptions and Theorem 1.1 that for any solution $z(t) = (q(t), \dot q(t))$ of Eq. (2.1) and the set $J \subset (1,\dots,k_*)$, we have the inclusion $\Lambda^+(z) \subset E(w_J = 0)$.

Theorem 2.1. Suppose that for some set $J \subset (1,\dots,k_*)$ there is a finite set of locally Lipschitz functions $V_i(q,\dot q)$ $(i = 1, 2, \dots, N)$ for which

$$(J \setminus N(\dot q) \neq \emptyset) \wedge (\forall j \in J \setminus N(\dot q),\ |N_j| = 0) \Rightarrow (\exists i \in \overline{1, N}, V_i = 0, D^{*+}V_i \neq 0) \tag{2.4}$$

Then $\Lambda^+(z) \subset H_J$ for any solution $z(t)$ of Eq. (2.1).

The proof follows from (2.3), (2.4) and Theorem 1.2.

Theorem 2.2. Suppose that condition (2.4) is satisfied for the set $J \subset (1,\dots,k_*)$. Then for any solution $z(t)$ of Eq. (2.1) we have $\Lambda^+(z) \subset M_J$, and:

1. either $\|z(t)\| \to \infty$ or $M_J \neq \varnothing$ and $z(t)$ tends weakly to M_J;
2. either the solution $z(t)$ is unbounded, or $M_J \neq \varnothing$ and $z(t)$ tend to M_J.

The proof follows from Theorem 2.1, Lemma 1.2 on the semi-invariance of the set $\Lambda^+(z)$, the definitions of the set M_J and the generalized friction forces Q_s^T.

Theorem 2.3. Suppose that for the set $J = (1,\dots,k_*)$ condition (2.4) is satisfied and the dissipation is total with respect to $\dot q^{k^*+1},\dots,\dot q^k$, that is

$$\sum_{i=1}^{k} D_i(q,\dot q)\dot q^i \leqslant -\gamma \sum_{i=k_*+1}^{k} \dot q^{i2} \tag{2.5}$$

for some $\gamma > 0$. Then for any solution $z(t)$ of Eq. (2.1), $\Lambda^+(z) \subset M$, and

1. either $\|z(t)\| \to \infty$ or $M \neq \varnothing$ and $z(t)$ tends weakly to M;
2. either the solution $z(t)$ is unbounded, or $M \neq \varnothing$ and $z(t)$ tends to M;
3. $M \neq \varnothing \Leftrightarrow \|z(t)\| \to \infty$ for any solution $z(t)$ of Eq. (2.1).

Proof. We put $w(q,\dot q) = -D^+V_0(q,\dot q)$. Then $\Lambda^+(z) \subset E(w = 0)$ and it follows from (2.5) that $\dot q^i = 0$ for all $i = k_* + 1,\dots,k$, $(q,\dot q) \in \Lambda^+(z)$. The theorem now follows from Theorem 2.2 and Lemma 1.2 on the semi-invariance of the set $\Lambda^+(z)$.

3. EXAMPLE

Consider a plane mechanical system consisting of a piston which moves with friction along an inclined rectilinear pipe, and a heavy absolutely rigid body which rotates with friction about a cylindrical hinge mounted on the piston (for a detailed description see [12]).

The equations of motion of the system in Lagrange form, after transformation, can be written in the form

$$\begin{aligned} &m\ddot{x} + m_2 r\cos\beta\ddot{\beta} = m_2 r\dot{\beta}^2\sin\beta - mg\sin\alpha + Q_1^T \\ &m_2 r\cos\beta\ddot{x} + J\ddot{\beta} = -m_2 gr\sin(\alpha+\beta) + Q_2^T \end{aligned} \tag{3.1}$$

The moduli of the normal reactions have the form

$$|N_1| = |m_2 r(\ddot{\beta}\sin\beta + \dot{\beta}^2\cos\beta) + mg\cos\alpha|$$

$$|N_2| = m_2[(\ddot{x} + r\ddot{\beta}\cos\beta - r\dot{\beta}^2\sin\beta + g\sin\alpha)^2 + (r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g\cos\alpha)^2]^{1/2}$$

The generalized friction forces under equilibrium with respect to each of the generalized coordinates x, β can be written as

$$Q_1^{T0} = m_2 r(\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta) + mg\sin\alpha \quad (\dot{x}=0, \ddot{x}=0)$$

$$Q_2^{T0} = m_2 r(\ddot{x}\cos\beta + g\sin(\alpha+\beta)) \quad (\dot{\beta}=0, \ddot{\beta}=0)$$

In the general case, the generalized friction forces are defined by the above rule for s = 1, 2, generalized coordinates $q^1 = x$, $q^2 = \beta$, velocities $\dot{q}^1 = \dot{x}$, $\dot{q}^2 = \dot{\beta}$ and accelerations $\ddot{q}^1 = \ddot{x}$, $\ddot{q}^2 = \ddot{\beta}$.

The inequalities 6.4 of [12] give sufficient conditions for the assumptions of Section 1 to be satisfied for Eqs (3.1).

The set of equilibrium positions for system (3.1) will be

$$M = \{(q,\dot{q}): \dot{x}=0,\ \dot{\beta}=0,\ f_1 \geqslant \operatorname{tg}\alpha,\ f_2 \geqslant r|\sin(\alpha+\beta)|\}$$

where f_1, f_2 are the friction coefficients (constant quantities) in the piston and hinge, respectively. We take the basic Lyapunov function as the energy of the system

$$V_0 = T + \Pi = \tfrac{1}{2}(m\dot{x}^2 + 2m_2 r\dot{x}\dot{\beta}\cos\beta + J\dot{\beta}^2) + mgx\sin\alpha + m_2 gr(1-\cos(\alpha+\beta))$$

For the set of indices J_0 = {1, 2} we consider the auxiliary Lyapunov functions

$$V_1 = \dot{x},\ V_2 = \dot{\beta},\ V_3 = r\dot{\beta}^2 + g\cos(\alpha+\beta),\ V_4 = \sin(\alpha+\beta)$$

and the function

$$w = f_1|N_1||\dot{x}| + f_2|N_2||\dot{\beta}| = -D^+V_0$$

We will show that condition (2.4) holds for system (3.1). The conditions $(J_0\backslash N(\dot{q}) \neq \varnothing) \wedge (\forall j \in J_0\backslash N(\dot{q}), |N_j| = 0)$ can be satisfied in one of the following three ways

1) $N(\dot{q}) = \{2\}, |N_1| = 0\ (\dot{\beta} = 0, \dot{x} \neq 0)$

2) $N(\dot{q}) = \{1\}, |N_2| = 0\ (\dot{\beta} \neq 0, \dot{x} = 0)$

3) $N(\dot{q}) = \varnothing, |N_1| = 0, |N_2| = 0\ (\dot{\beta} \neq 0, \dot{x} \neq 0)$

It is easy to see that the functions $|N_1|$ and $|N_2|$ cannot both vanish at the same time, and thus Case 3 is impossible under any conditions. Consider Cases 1 and 2.

1. If $D^+\dot{\beta} = 0$, from the conditions $|N_1| = 0$, $\dot{\beta} = 0$ we obtain $mg\cos\alpha = 0$, which is impossible (since $0 \leqslant \alpha < \pi/2$). Hence, $V_2 = 0$, $D^+V_2 \neq 0$.

2. If $D^+\dot{x} \neq 0$, then $V_1 = 0$ and $D^+V_1 \neq 0$. Let $D^+\dot{x} = 0$. Then from the condition $|N_2| = 0$ we obtain

$$\begin{aligned} &r\ddot{\beta}\cos\beta - r\dot{\beta}^2\sin\beta + g\sin\alpha = 0 \\ &r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g\cos\alpha = 0 \end{aligned} \tag{3.2}$$

Multiplying the first of Eqs (3.2) by $\sin\beta$, the second by $\cos\beta$, and then subtracting the first equation of (3.2) from the second, we obtain $r\dot{\beta}^2 + g\cos(\alpha+\beta) = 0$, and thus $V_3 = 0$.

From the second equation of (3.1) under the condition $\ddot{x} = 0$, $|N_2| = 0$, we find $\ddot{\beta} = -m_2 gr\sin(\alpha + \beta)/J$, whence it follows that $D^+V_3 = -\dot{\beta}g\sin(\alpha + \beta)(2r^2m_2/J+1)$. Thus, $D^+V_3 \neq 0$ if $\sin(\alpha + \beta) \neq 0$.

But if $\sin(\alpha + \beta) = 0$, then $V_4 = 0$ and $D^+V_4 = \cos(\alpha + \beta)\dot{\beta} \neq 0$. This completes the investigation of the conditions of Theorem 2.3 for Eqs (3.1). It can be concluded that system (3.1) is dichotomous.

We will make two concluding comments: if $\operatorname{tg}\alpha > f_1$, then $M = \varnothing$ and there are no bounded solutions in system (3.1) (to be more precise, all the solutions are infinitely large); if $\operatorname{tg}\alpha = f_1$, then $M = \varnothing$, but Eqs (3.1) have a solution (which is unbounded)

$$x = \dot{x}_0 t + x_0, \quad \dot{x} = \dot{x}_0, \quad \beta = \beta_0, \quad \dot{\beta} = 0$$

where $f_2 \geqslant r|\sin(\alpha + \beta_0)|$, $\dot{x}_0 < 0$, which does not tend to M even weakly.

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